# Math 222A Lecture 8 Notes

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## 1 Expressing Hamilton-Jacobi Equations in Terms of Calculus of Variations

### 1.1 Recap: Hamilton-Jacobi equations

Last time, we started talking about Hamilton-Jacobi equations, as an example of first order PDEs:

$$\begin{cases} u_t + H(x, Du) = 0 \\ u(0) = u_0 \end{cases}$$

The characteristics for this system were given by

$$\begin{cases} \dot{u} = H_p(x, p) \\ \dot{p} = -H_x(x, p) \\ \dot{z} = H_p(x, p) \cdot p - H(x, p) \end{cases}$$

with initial data

$$\begin{cases} x(0) = x_0 \\ p(0) = \partial_x u_0. \end{cases}$$

The equations for $\dot{u}$ and $\dot{p}$ are called the **Hamilton equations**. We noticed that we only need to solve them first to get the characteristics, and then we can integrate the $\dot{z}$ equation to solve it after the fact.

### 1.2 Calculus of variations

Today, we will be looking at the calculus of variations. Here is the setup: We have a function $L(x, q)$ we call the **Lagrangian**, and to each function $x : [0, T] \to \mathbb{R}$, we associate to this function an **action functional**

$$\mathcal{L}(x) = \int_0^T L(x, \dot{x}) \, dt.$$

The question we want to ask is: what are the minimizers of $\mathcal{L}$? We are looking for

$$\min_{x:[0,T]\to\mathbb{R}} \mathcal{L}(x).$$

We can think of $\mathcal{L}$ giving the cost of the trajectory $x$. So we want to find the most efficient trajectory $x$.

If we were just minimizing a function in $\mathbb{R}^n$, we would look for critical points. In particular, for $f : \mathbb{R}^n \to \mathbb{R}$, a minimum point in a critical point if $\nabla f = 0$. How do we do this in the case of our functional? We can talk in terms of directional derivatives. Replace $x$ by $x + hy$ and look at the map $h \mapsto \mathcal{L}(x + hy)$, where $h = 0$ is a minimum point. Assume that our perturbation $y$ is compactly supported. In this case, at $h = 0$, we have

$$\begin{aligned} 0 &= \frac{d}{dh}\mathcal{L}(x+hy) \\ &= \frac{d}{dh}\int_0^T L(x+hy, \dot{x}+h\dot{y})\,dt \\ &= \int_0^T L_x(x,\dot{x})\cdot y + L_q(\dot{x})\cdot \dot{y}\,dt, \end{aligned}$$

where we are using $q$ as a placeholder for the second variable, as we did with $p$ before. This holds for all $y \in C_0^\infty([0,T])$. To deal with the $\dot{y}$ term, we integrate by parts (using the compact support assumption):

$$= \int_0^T y(L_x(x,\dot{x}) - \frac{d}{dt}L_q(x,\dot{x}))\ dt$$

when integrated against any function with compact support, the part inside the parentheses gives 0. So it must equal 0, Thus, we have actually proven a theorem:

**Theorem 1.1** (Euler-Lagrange equation)**.** *$x$ is a critical point for $\mathcal{L}$ if and only if it solves*

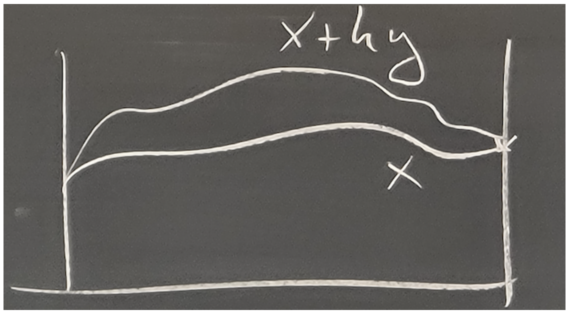
$$L_x(x,\dot{x}) - \frac{d}{dt}L_q(x,\dot{x}) = 0.$$

**Remark 1.1.** The PDE analogue takes a function $u : \mathbb{R}^n \to \mathbb{R}$ and gives the Euler-Lagrange equation

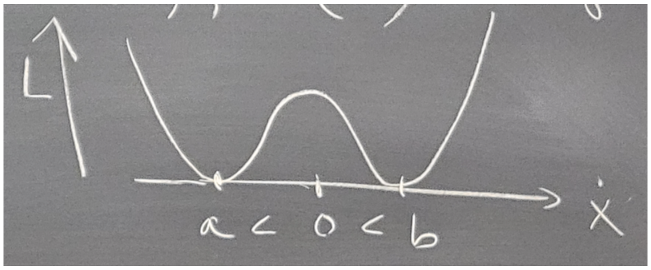
$$L_x(u,\partial u) - \partial_j L_{q_j}(u,\partial u) = 0,$$

which is a second order PDE.

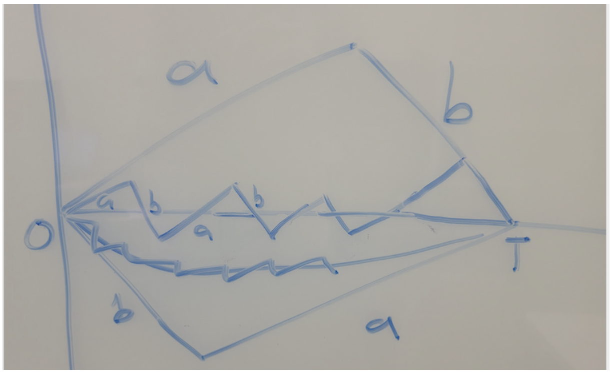
**Remark 1.2.** Our perturbation does not change the values at the endpoints $x(0), x(T)$, so it gives critical points in a context where $x(0)$ and $x(T)$ are fixed.




**Remark 1.3.** Suppose $L = L(\dot{x})$ is the following "double well potential."

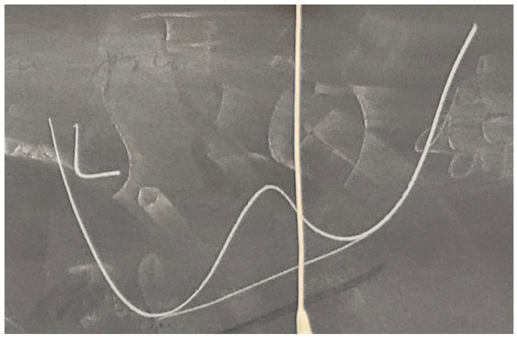



Suppose also that $x(0) = x(T)$. We want to minimize $\int_0^T L(\dot{x})\,dt \geq 0$. Can we achieve 0? We can make a line with slope $a$ and then a line with slope $b$ to get 0 as the minimum (notice that this is not differentiable!). Alternatively, we can alternate between lines of slope $a$ and $b$ in any number of ways as follows:

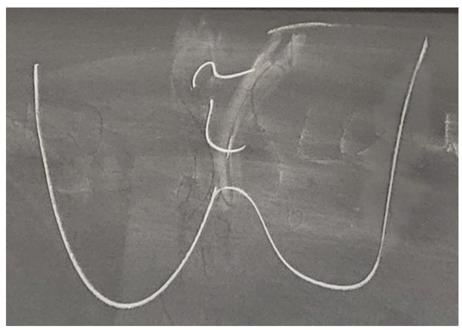



So we get that the infimum is 0 (since we can approximate any piecewise function by smoothing out the corners), and the minimum is 0 if we allow for any Lipschitz function $x$. In fact, all trajectories with slopes between $[a, b]$ are limiting minimizers. This means we are actually dealing with an **effective Lagrangian** $L_{\text{eff}}$ with the hump between $a$ and $b$ flattened out. The effective Lagrangian $L_{\text{eff}}$ is the **convex envelope** of $L$.

If we had another Lagrangian like the following, could we again look at the convex envelope?




Suppose we add a linear constant to get $\widetilde{L}(q) = L + c \cdot q$. Then we get the following picture, which is the same as before:



So the effective Lagrangian must be convex as a function of $q$. For PDEs, convexity is no longer required. Instead, we require **rank one convexity**, which is given by convexity in one variable at a time.

**Example 1.1.** Here is an example that comes from classical mechanics. Suppose we have a particle with trajectory $x(t)$ moving in a conservative force field $F = \nabla\phi$, where $\phi$ is the potential. Then we have the Lagrangian

$$L(x, q) = \underbrace{\frac{1}{2}mq^2}_{\text{kinetic energy}} - \underbrace{\phi(x)}_{\text{potential energy}},$$

where we have $\phi_x = \frac{d}{dt}(m\dot{x})$, which we can write as $m \cdot \ddot{x} = F(x)$, which is Newton's law.

## 1.3 Connecting the Hamilton-Jacobi equations to the Euler Lagrange equations

Returning to Hamilton-Jacobi equations, we have $x, p$ with the function $H$, and we want to relate this to the $x, q = \dot{x}$ and $L$ in the Euler-Lagrange equation. We can think of the Euler-Lagrange equation as a system for $x$ and $q$ via

$$\begin{cases} \dot{x} = q \\ \frac{d}{dt} L_q(x, q) = L_x. \end{cases}$$

We want to let $p = L_q(x, q)$. For this to make sense, we need $q \mapsto L_q(x, q)$ to be a diffeomorphism from $\mathbb{R}^n \to \mathbb{R}^n$ for fixed $x$.

**Proposition 1.1.** *If $L : \mathbb{R}^n \to \mathbb{R}$ is strictly convex and **coercive** (meaning $\lim_{q \to \infty} \frac{L(q)}{|q|} = \infty$), then $q \mapsto L_q$ is a diffeomorphism.*

*Proof.* Injectivity: $L$ is strictly convex, so the graph of $L$ is above its tangent lines at points of nonintersection:

$$L(y) > L(x) + (y - x) DL(x), \qquad y \neq x.$$

We can use this to write

$$(y - x)(DL(y) - DL(x)) > 0, \qquad y \neq x.$$

This gives injectivity.

Surjectivity: We want to minimize $L(x, q) - p \cdot q$. If a minimum exists, then the gradient must equal 0:

$$L_q(x, q) = p,$$

which is our surjectivity. Why must the minimum exist? This is because $\lim_{q \to \infty} L(x, q) - p \cdot q = \infty$ by coercivity.

To check that this is a local diffeomorphism, the differential of $q \mapsto L_q(x, q)$ is $L_{qq} \geq 0$. In fact, by strict convexity, this is $> 0$. $\square$

So we have $p = L_q(x, q)$. We will define $H(x, p) = \max_q p \cdot q - L(x, q)$, Note that this is the same quantity we dealt with in the above proof. The functions $p \cdot q - L(x, q)$ are linear in $p$, so this maximum is convex.

**Proposition 1.2.** *$H$ is convex and coercive.*

*Proof.* This comes from the strict convexity and coercivity of $L$. $\square$

**Proposition 1.3.**

$$q = H_p(x, p).$$

*Proof.* This is a maximum, so $H(p) + L(q) - pq \leq 0$, with equality if $p = L_q(x, q)$. Now fix $q$ and vary $p$! Then $p$ is a maximum point for this expression when the derivative $H_p(p) - q = 0$. $\square$

Now let's change our variables: The Euler-Lagrange equations say

$$L_x(x, q) - \frac{d}{dt}\underbrace{L_q(x, q)}_{p} = 0$$

So we get

$$\begin{cases} \dot{p} = L_x(x, q) \stackrel{?}{=} -H_x(x, p) \\ \dot{x} = q = H_p(x, p). \end{cases}$$

We have

$$H(x, p) + L(x, q) - p \cdot q \leq 0,$$

If we think of $p = p(x, q)$, we can take $\frac{d}{dx}$ to get

$$H_x(x, p) + L_x(x, q) + \underbrace{(H_p(x, p) - q)}_{=0} \cdot \frac{\partial p}{\partial q} = 0.$$

So this gives us our relationship between $H_x$ and $L_x$.